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A Ferromagnetic Inequality

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ABSTRACT

The investigation of the monotonicity of correlation as a function on interaction in Ising ferromagnets with integral spin led to the inequality on subgroups and cosets. The mathematics seems similar to that arising in error-correcting codes.

THEOREM. *Let φ be a real valued function on the non-negative integers. A necessary and sufficient condition that for each positive integer n, each finite group G, subgroups $G_1, \dots, G_n$ and their left cosets $g_1G_i, \dots, g_nG_n$*

$$\sum_{g\in G} \varphi(\#\{i : g \in G_i\}) \geqslant \sum_{g\in G} \varphi(\#\{i : g \in g_iG_i\}) \tag{1}$$

is that

$$\Delta^k\varphi(0) \geqslant 0, \qquad k = 2, 3, \dots . \tag{2}$$

PROOF: Let $N =_{df} \{1, \dots, n\}$. $(\forall\, S \subset N)$ $G_S =_{df} \bigcap_{i\in S} G_i$, $G^S =_{df} \bigcap_{i\in S} g_iG_i$ $\therefore$ $G_\phi = G^\phi = G$. Since G^S is empty or a coset of G_S, $\#G^S = 0$ or $\#G^S = \#G_s$, and in any case

$$\#G^S \leqslant \#G_S. \tag{3}$$

Let $s_g =_{df} \#\{i : g \in G_i\}$ and $s^g =_{df} \#\{i : g \in g_iG_i\}$.

In order to prove sufficiency assume (2). It is helpful to consider the special case $n = 2$. From the separate examination of those g such that $s_g = 2, 1$, and 0 one obtains

$$\begin{aligned}\sum_{g\in G} \varphi(s_g) &= \#G_{\{1,2\}}\varphi(2) + (\#G_1 - \#G_{\{1,2\}})\varphi(1) + (\#G_2 - \#G_{\{1,2\}})\varphi(1)\\ &\quad + (\#G - \#G_1 - \#G_2 + G_{\{1,2\}})\varphi(0)\\ &= (\#G_{\{1,2\}})\Delta^2\varphi(0) + (\#G_1 + \#G_2)\Delta\varphi(0) + (\#G)\varphi(0).\end{aligned}$$

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Analogously

$$\sum_{g \in G} \varphi(s^g) = (\#G^{\{1,2\}})\Delta^2\varphi(0) + (\#G^{\{1\}} + \#G^{\{2\}})\Delta\varphi(0) + (\#G)\varphi(0).$$

Application of (3) and $\#G^{\{i\}} = \#G_i$ yields (1) for the case $n = 2$.

For the case of $n \geqslant 2$ one uses the method of inclusion and exclusion to get

$$\begin{aligned}
\sum_{g \in G} \varphi(s_g) &= \sum_{L \subset N} \sum_{L \subset M \subset N} \#G_M(-1)^{\#M - \#L}\varphi(\#L) \\
&= \sum_{M \subset N} \#G_M \sum_{L \subset M} (-1)^{\#M - \#L}\varphi(\#L) \\
&= \sum_{M \subset N} \#G_M \Delta^{\#M}\varphi(0).
\end{aligned}$$

Analogously

$$\sum_{g \in G} \varphi(s^g) = \sum_{M \subset N} \#G^M \Delta^{\#M}\varphi(0).$$

Application of (3) and $\#G^{\{i\}} = \#G_i$ now yields (1) to complete the proof of sufficiency.

For the proof of the necessity first consider the case $G_1 = G_2$, $g_1G_1 \cap G_2 = \phi$, $g_2G_2 = G_2$, and $n = 2$. Here $\#G^N = 0$ and the difference of the two sides of (1) is $\#G_N\Delta^2\varphi(0)$. $\therefore\ \Delta^2\varphi(0) \geqslant 0$.

For the next cases we consider G as the additive group of vector space V of dimension k over Z_2. Subscripts and superscripts on V have the same meanings as those on G.

Let $x_1, x_2, \ldots, x_{n+1}$, $2 \leqslant n$, be independent vectors in V a vector space over Z_2. Let V_j be the vector space spanned by

$$\{x_{n+1} + x_1, \ldots, \widehat{x_{n+1} + x_j}, \ldots, x_{n+1} + x_n\}, \qquad 1 \leqslant j \leqslant n,$$

and let V_{n+1} be spanned by $\{x_1, \ldots, x_n\}$. Let $V^j = V_j$, $1 \leqslant j \leqslant n$, and $V^{n+1} = x_{n+1} + V_{n+1}$. Now $V_N = \{0\}$ and, if $\phi \neq M \subset N$ and $M \neq N$, then $V_M = V^M$ and $i \in N \backslash M \Rightarrow x_{n+1} + x_i \in V_M$. Thus $V^M \cap V^{n+1} \neq \phi$ but $V^N \cap V^{n+1} = \phi$. Therefore the difference of both sides in (1) becomes $\#V^{N \cup \{n+1\}}\Delta^{n+1}\varphi(0) \geqslant 0$ and $\Delta^{n+1}\varphi(0) \geqslant 0$, $n = 2, 3, \ldots$, which completes the proof of the theorem.

It should be observed that the theorem continues to be valid if G merely ranges over the additive groups of finite dimensional vector spaces over some galois field. On the other hand, if in the hypothesis the restriction $n \leqslant n_0$ were introduced then the restriction $k = 2, 3, \ldots, n_0$ should be introduced into (2).

For the application to the spin-s Ising ferromagnet,

$$\varphi(n) = ((2s + 1)/2s)^n,$$

so that (2) and (1) apply.